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LETTER TO THE EDITOR

Fujimoto–Watanabe equations and differential substitutions

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Abstract. New Fujimoto–Watanabe non-constant separant evolution equations admitting generalized symmetries are shown to be connected with the Korteweg–de Vries, Sawada–Kotera, Kaup and linear equations via chains of differential substitutions. A conjecture is proposed which explains why the Ibragimov transformation to constant separant equations is a universal link in those chains.

Recently, Fujimoto and Watanabe [1] classified third-order polynomial evolution equations of uniform rank with non-constant separants, which admit generalized symmetries [2] (non-trivial Lie–Bäcklund algebras [3]), and found eight equations of that kind:

$$u_t = u_1^3 u_3 + \alpha u_1^4 \tag{1}$$

$$u_t = u_1^3 u_3 + \alpha u_1^3 \tag{2}$$

$$u_t = u^3 u_3 + 3u^2 u_1 u_2 + \alpha(u^3 u_2 + u^2 u_1^2) + \tfrac{2}{9}\alpha^2 u^3 u_1 \tag{3}$$

$$u_t = u^3 u_3 + 3u^2 u_1 u_2 + 4\alpha u^3 u_1 \tag{4}$$

$$u_t = u^3 u_3 + 3u^2 u_1 u_2 + 3\alpha u^2 u_1 \tag{5}$$

$$u_t = u^3 u_3 + \alpha u^3 u_1 \tag{6}$$

$$u_t = u^3 u_3 + \tfrac{3}{2}u^2 u_1 u_2 + \alpha(u^3 u_2 + u^2 u_1^2) + \tfrac{2}{9}\alpha^2 u^3 u_1 \tag{7}$$

$$u_t = u^3 u_3 + \tfrac{3}{2}u^2 u_1 u_2 + \alpha u^2 u_1 \tag{8}$$

where $u = u(x, t)$, $u_t = \partial u/\partial t$, $u_k = \partial^k u/\partial x^k$, α is a parameter. Fujimoto and Watanabe found also two fifth-order equations

$$u_t = u^5 u_5 + 5u^4(u_1 u_4 + 2u_2 u_3) \tag{9}$$

$$u_t = u^5 u_5 + 5u^4(u_1 u_4 + \tfrac{1}{2}u_2 u_3) + \tfrac{15}{4}u^3 u_1^2 u_3 \tag{10}$$

which also admit generalized symmetries and do not belong to hierarchies of equations (1)–(8). Fujimoto and Watanabe pointed out that (2) and (1) are the potential forms of (5) and (4), the last two being equivalent to the Korteweg–de Vries (KdV) equation and the modified KdV equation, respectively, via the Ishimori transformation defined as $x = \int u \, dx$ in [1].

In this letter, we construct chains of differential substitutions which connect the Fujimoto–Watanabe equations (1)–(10) with the KdV, Sawada–Kotera [4], Kaup [5] and linear equations. We consider (1)–(10), including equations transformed into the KdV equation by Fujimoto and Watanabe, in order to demonstrate and generalize certain common features of the chains of differential substitutions involved. Proofs, which can be restored easily, are omitted here.

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Equations (3)-(10) have the form $u_t = u^n u_n + \ldots$, $n = 3$ or 5. As one can prove, any differential substitution (if it exists) connecting equations $u_t = u^n u_n + \ldots$ and $v_t = v_n + \ldots (n > 1)$ has the form $x = b$, $u = D_y b$, where $b(y, v, v_1, \ldots, v_m)$ is a certain function, $v = v(y, t)$, $v_k = \partial^k v/\partial y^k$, D_y is the total y-derivative. Such a substitution with $b = v$ was introduced by Ibragimov [3, 6] as a transformation between the Harry Dym equation and modified via the Schwarzian derivative KdV equation (respectively (6) above and (16) below at $\alpha = 0$). The Ibragimov transformation $x = v$, $u = v_1$ can be proven to map only the equations

$$v_t = \xi y v_1 + v_1 a(v, v_1, v_1^{-1} v_2, \ldots, (v_1^{-1} D_y)^{n-1} v_1) \tag{11}$$

into the corresponding ones

$$u_t = \xi u + u^2 D_x a(x, u, u_1, \ldots, u_{n-1}) \tag{12}$$

with arbitrary parameter ξ, function a and order n (the proof is based on the technique of [7, 8]). Equations (3)-(10) actually have the form $u_t = u^2 D_x(\ldots)$. This is a surprising phenomenon but not a chance one. Indeed, any equation $u_t = f(x, u, u_1, \ldots, u_n)$ admitting a generalized symmetry of order higher than n has the conserved density $(\partial f/\partial u_n)^{-1/n}$ [3]. If $f = u^n u_n + \ldots$, then $(u^{-1})_t + D_x(\ldots) = 0$ takes place, and $f = u^2 D_x(\ldots)$ turns out to be necessary. Therefore (3)-(10) belong to class (12), and corresponding equations of class (11) are as follows:

$$v_t = v_3 + \alpha v_1 v_2 + \tfrac{1}{9}\alpha^2 v_1^3 \tag{13}$$

$$v_t = v_3 + 2\alpha v_1^3 \tag{14}$$

$$v_t = v_3 + 3\alpha v_1^2 \tag{15}$$

$$v_t = v_3 - \tfrac{3}{2} v_1^{-1} v_2^2 + \tfrac{1}{2}\alpha v_1^3 \tag{16}$$

$$v_t = v_3 - \tfrac{3}{4} v_1^{-1} v_2^2 + \alpha v_1 v_2 + \tfrac{1}{9}\alpha^2 v_1^3 \tag{17}$$

$$v_t = v_3 - \tfrac{3}{4} v_1^{-1} v_2^2 + \alpha v_1^2 \tag{18}$$

$$v_t = v_5 - 5 v_1^{-1} v_2 v_4 + 5 v_1^{-2} v_2^2 v_3 \tag{19}$$

$$v_t = v_5 - 5 v_1^{-1} v_2 v_4 - \tfrac{15}{4} v_1^{-1} v_3^2 + \tfrac{65}{4} v_1^{-2} v_2^2 v_3 - \tfrac{135}{16} v_1^{-3} v_2^4. \tag{20}$$

The correspondence under the Ibragimov transformation $x = v$, $u = v_1$ is $(k+10) \to (k)$, $k = 3, 4, \ldots, 10$. In the right-hand sides of (13)-(20) we omit the trivial term βv_1 caused by the invariance of (12) under $a \to a + \beta$, β is any constant.

Now, constant separant equations (13)-(20) can be reduced to more familiar ones. Third-order equations (13)-(18) fall under Habirov's classification [9] which provides the following: (13) and (17) are mapped into the linear equation $w_t = w_3$ via $w = \exp(\tfrac{1}{3}\alpha v)$ and $w = (\tfrac{2}{3}\alpha v_1)^{1/2} \exp(\tfrac{1}{3}\alpha v)$ respectively; (14), (15), (16) and (18) are mapped onto the KdV equation $w_t = w_3 + 3 w w_1$ via $w = 2(-\alpha)^{1/2} v_2 + 2\alpha v_1^2$, $w = 2\alpha v_1$, $w = v_1^{-1} v_3 - \tfrac{3}{2} v_1^{-2} v_2^2 + \tfrac{1}{2}\alpha v_1^2$ and $w = (-\tfrac{1}{3}\alpha)^{1/2} v_1^{-1/2} v_2 + \tfrac{2}{3}\alpha v_1$ respectively; $w = w(y, t)$, $w_k = \partial^k w/\partial y^k$. Fifth-order equations (19) and (20) are mapped onto the Kaup equation [5] $w_t = w_5 + 10 w w_3 + 25 w_1 w_2 + 20 w^2 w_1$ and Sawada-Kotera equation [4] $w_t = w_5 + \tfrac{5}{3} w w_3 + \tfrac{5}{2} w_1 w_2 + \tfrac{5}{4} w^2 w_1$, respectively, via $w = v_1^{-1} v_3 - \tfrac{3}{2} v_1^{-2} v_2^2$ [7] which is the Schwarzian derivative. *Equation (19) is also mapped onto the Sawada-Kotera equation* via the differential substitution $w = -2 v_1^{-1} v_3$ associated with the inverse scattering transform technique [10] at zero spectral parameter.

The remaining equations (1) and (2) admit the map $z=u_1$ into (4) and (5) on z respectively [1]. Note that the form $z=(\partial f/\partial u_n)^{1/n}$ is necessary for any differential substitution which maps an equation $u_t=f(x, u, u_1, \ldots, u_n)$ onto $z_t=z^n z_n+\ldots$. Does this substitution exist for every non-constant separant evolution equation admitting a generalized symmetry? Is the Ibragimov transformation applicable to the resulting equation $z_t=z^n z_n+\ldots$? Do the involved transformations preserve generalized symmetries? We believe the following conjecture answers these questions correctly.

Conjecture. If two non-constant separant evolution equations $u_t=f(x, u, u_1, \ldots, u_n)$ and $u_s=g(x, u, u_1, \ldots, u_m)$ $(m>n>1)$ are compatible, i.e. $\Sigma_k (\partial f/\partial u_k)D_x^k g \equiv \Sigma_k (\partial g/\partial u_k)D_x^k f$, then: (i) these equations admit the differential substitution $z=(\partial f/\partial u_n)^{1/n} \equiv (\partial g/\partial u_m)^{1/m}$ (scaling of s is implied); (ii) the resulting equations $z_y=z^n z_n+\ldots$ and $z_s=z^m z_m+\ldots$ are compatible and belong to class (12); (iii) the corresponding equations $v_t=v_n+\ldots$ and $v_s=v_m+\ldots$ of class (11) are compatible too.

At present, we have a fragmentary proof only, and the conjecture should be used as certain tactical means. In addition to the Fujimoto–Waranabe equations, let us quote one more example illustrating the conjecture in action. The equation

$$u_t=D_x(u^{-3}u_2-3u^{-4}u_1^2)+1 \tag{21}$$

admits an infinite algebra of generalized symmetries which is destroyed by the transformation $y=u$, $w=u_1$ [11]. Our conjecture indicates transformations which preserve generalized symmetries and connect (21) with the KdV equation. Indeed, the separant of (21) is u^{-3}, the substitution $z=u^{-1}$ gives $z_t=z^2D_x(zz_2+z_1^2-x)$ which belong to class (12), and the corresponding equation of class (11) is $v_t=v_3-vv_1$.

Regardless of the validity of our conjecture, differential substitutions need to be used in any classification of nonlinear partial differential equations. Anyhow, every new remarkable equation must be suspected of being an appropriately spoilt well known one.

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